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A Bankruptcy Problem and an Information Trading Problem: Applications to k-Convex Games

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Abstract: The paper is devoted to two real problems that generate a cooperative game model with a so-called k-convex characteristic function when certain conditions are fulfilled. Both a bankruptcy problem and an information trading problem are modelled as a cooperative game by constructing the corresponding bankruptcy game as well as the information market game. Firstly, it is established that the bankruptcy game is a k-convex n-person game where $1 \leq k \leq n-2$ if and only if the estate is sufficient to meet the claims of creditors in any $(n-k)$-person coalition. Secondly, it is shown that the k-convexity property for the information market game is equivalent to the nonexistence of profits with respect to a restricted class of submarkets.

Key words: Cooperative game, convex game, k-convex game

1 Introduction

Shapley (1971) started the study of the notion of convexity for cooperative games in characteristic function form. A cooperative game is called a convex game if its characteristic function is a supermodular function on the set of all coalitions (subsets of a given player set). Shortly, the supermodularity of a real-valued set-function means that the incentives of an arbitrarily fixed coalition for joining another disjoint coalition don't decrease as the coalition grows: the so-called "snowballing" or "bandwagon" effect.

The application of game theoretic analysis to a variety of real problems has resulted in the discovery of at least three problems that generate a cooperative game model with a supermodular characteristic function. First of all, Littlechild and Owen (1973) treated the problem of setting airport landing charges for different types of aircraft. Their game theoretic approach to the airport cost allocation problem is based on the construction of the characteristic cost function of the so-called airport cost game. Secondly, Topkis (1983) considered the problem of selecting from the set of available activities a subset of public and private activities that is optimal with respect to a specific net return function.

Topkis formulated the activity selection problem as a game theoretic problem by interpreting the maximum net return function as the characteristic function of the so-called activity selection game. Thirdly, O'Neill (1982) paid attention to the division problem of the entire estate of a deceased person among a number of creditors. O'Neill started the game theoretic approach to the bankruptcy problem by constructing the characteristic function of the so-called bankruptcy game. The airport cost allocation problem, the activity selection problem as well as the bankruptcy problem are modelled as a cooperative game where the associated characteristic function is supermodular. In other words, the induced airport savings game, the activity selection game and the bankruptcy game, all of them are convex games.

In the eighties and in the beginning of the nineties several generalizations of the notion of convexity for cooperative games have been proposed. For instance, Granot and Huberman (1982) introduced the notion of permutational convexity as an adjunct to their study of the problem of cost allocation among users of a minimum cost spanning tree network. In addition, Sharkey (1982) defined the notion of subconvexity. Both Sprumont (1990) and Iñarra and Usategui (1993) introduced, independently of each other, the same notion of average convexity. All three notions can be regarded as refinements in the sense that any convex game is permutationally convex, subconvex, and average convex.

Driessen (1986a,b) introduced the notion of k-convexity as another type of modification of the notion of convexity. For an arbitrary natural number k, an n-person game is said to be k-convex if a specifically induced cover game is a convex n-person game. In case $k = n$, then the induced game coincides with the original game and as such, the notion of n-convexity agrees with the notion of convexity. An extensive treatment of the game theory part involving k-convexity can be found in Driessen (1988, Chapter VII). From the game theoretic viewpoint, the k-convexity of a game suffices to obtain interesting properties for solutions of the game involved (particularly, the core and, to a less extent, the τ-value and nucleolus).

Up to the present there exists almost no literature on k-convex n-person games which arise from a real problem. The purpose of the paper is to present a first attempt to provide literature on k-convex n-person games of that kind. Section 3 (4 respectively) deals with the study of the k-convexity of the bankruptcy (information market) game which arises from the bankruptcy (information trading) problem. Section 2 is devoted to the treatment of the notion of k-convexity for cooperative games.

2 Convexity and k-Convexity of Cooperative Games

Let N be a finite nonempty set whose elements are called **players**. Subsets of the player set N are called **coalitions** and let $\mathcal{N}$ denote the set of all coalitions, i.e.,

$\mathcal{N} := \{S; S \subset N\}$. A **cooperative game in characteristic function form** (or shortly, a game) is a function $v\colon \mathcal{N} \to \mathbb{R}$ such that $v(\emptyset) = 0$. The real-valued set-function v on $\mathcal{N}$ is called the **characteristic function** of the game. The **worth** $v(S)$ of $S \in \mathcal{N}$ in the game v represents the total amount of payoff that the coalition S can attain by mutual cooperation between its members. The number of players in the coalition S is denoted by $|S|$.

We say the game is a convex game if its characteristic function is a supermodular function. To be exact, the game v is said to be **convex** if the function $v\colon \mathcal{N} \to \mathbb{R}$ satisfies one of the following three equivalent conditions (cf. Shapley, 1971, or Ichiishi, 1981):

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \qquad \text{for all } S, T \subset N \tag{2.1}$$

$$v(S \cup R) - v(S) \leq v(T \cup R) - v(T) \qquad \text{for all } R, S, T \subset N \text{ such that } S \subset T \subset N - R \tag{2.2}$$

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \qquad \text{for all } i \in N \text{ and all } S, T \subset N \text{ such that } S \subset T \subset N - \{i\}\ . \tag{2.3}$$

Both the convexity conditions (2.2) and (2.3) express the so-called "snowballing" or "bandwagon" effect mentioned in the introduction. In other words, the characteristic function of a convex game satisfies nondecreasing marginal returns with respect to the coalition growth and this property conforms to the "nondecreasing returns to scale" associated with convex production functions in economics. In the remainder of the section we recall the notion of k-convexity for cooperative games.

Definition 2.1: Let v be a game with player set N consisting of n players. The corresponding **utopia function** $u^v\colon N \to \mathbb{R}$ and **gap function** $g^v\colon \mathcal{N} \to \mathbb{R}$ are given by

$$u^v(i) := v(N) - v(N - \{i\}) \quad \text{for all } i \in N\ , \tag{2.4}$$

$$g^v(S) := \sum_{j \in S} u^v(j) - v(S) \qquad \text{for all } S \subset N\ . \tag{2.5}$$

The utopia payoff $u^v(i)$ to player $i \in N$ represents the marginal return to player i for the formation of the grand coalition N. In the framework of the division of the total savings $v(N)$ of the grand coalition among the players, no player should be paid more than his utopia payoff because of the possible threat of the formation of any $(n-1)$-person coalition. For each coalition $S \subset N$, the gap $g^v(S)$ equals the difference of the total utopia payoff to the coalition and the worth of the coalition. It is usually supposed that the gap function is non-negative, i.e., $g^v(S) \geq 0$ for all $S \subset N$. Note that $g^v(\emptyset) := 0$. We interpret the gap function g^v as a "measure" of the coalitions' discontentedness caused by the utopia function u^v. Here the discontentedness of any coalition is always compared with that of the grand coalition.

The first main idea of the notion of k-convexity is to formulate somehow the relative discontentedness of a coalition in terms of the size of the coalition. We use a fixed natural number k to divide the coalitions into two types according to their size with respect to k. The first main idea expresses that coalitions with at least k players are at least as discontented as the grand coalition. Since the exact numerical data of the most discontented coalitions is considered to be of no importance, the game is changed in such a way that the discontentedness of coalitions with at least k players is reduced to the level of the discontentedness of the grand coalition, while the discontentedness of the remaining coalitions is not affected by the change of the game. The second main idea of the notion of k-convexity is to require the convexity condition for the induced game.

Definition 2.2: (cf. Driessen, 1988, pages 173–175). Let $k \in \mathbb{N}$ and let v be an n-person game with player set N. The corresponding n-person game v_k with player set N is given by

$$\begin{aligned} v_k(S) &:= v(S) && \text{if } |S| < k \\ &= \sum_{j \in S} u^v(j) - g^v(N) = v(N) - \sum_{j \in N-S} u^v(j) && \text{if } |S| \geq k \;. \end{aligned}$$

The n-person game v is said to be ***k*-convex** if

(i) $g^v(S) \geq g^v(N)$ for all $S \subset N$ with $|S| \geq k$ (2.6)

(ii) the corresponding game v_k is a convex game.

Let k be an arbitrary natural number. In view of (2.4) and (2.5), we always have $v_k(N) = v(N)$, $v_k(N - \{i\}) = v(N - \{i\})$ for all $i \in N$ and consequently, $u^{v_k}(i) = u^v(i)$ for all $i \in N$ as well as $g^{v_k}(S) = g^v(N)$ if $|S| \geq k$, $g^{v_k}(S) = g^v(S)$ if $|S| < k$. The latter formulas concerning the gap function are the realizations of the first main idea of the notion of k-convexity mentioned above Definition 2.2.

For any $k \geq n-1$, we observe that the two games v and v_k are equal, i.e., $v_k(S) = v(S)$ for all $S \subset N$, and in addition, the condition (2.6) is either superfluous or trivial because of $g^v(N - \{i\}) = g^v(N)$ for all $i \in N$. Therefore, for any $k \geq n - 1$ we conclude that an n-person game is k-convex if and only if the game itself is convex. Especially, the notion of n-convexity agrees with the notion of convexity. Throughout the remainder of the paper we suppose that the natural number k satisfies $1 \leq k \leq n - 2$.

In case a game v is convex, then the supermodularity condition (2.3) for the characteristic function $v: \mathcal{N} \to \mathbb{R}$ yields that the corresponding gap function $g^v: \mathcal{N} \to \mathbb{R}$ is a monotonic set-function, i.e., $g^v(S \cup \{i\}) \geq g^v(S)$ for all $i \in N$ and all $S \subset N - \{i\}$. In particular, the grand coalition N in a convex game v is the most discontented coalition because of $g^v(N) \geq g^v(S)$ for all $S \subset N$. Recall that coalitions with at least k players in a k-convex n-person game are at least as discontented as the grand coalition because of the condition (2.6). Thus, the equalities $g^v(S) = g^v(N)$ for all $S \subset N$ with $|S| \geq k$ are necessary for the k-convexity of a convex n-person game. In preparation for Section 3, we prove that these equalities in the condition (2.6) are also sufficient for the k-convexity of a convex game.

The relevant equalities in (2.6) imply that for all $S \subset N$ with $|S| \geq k$

$$v_k(S) = \sum_{j \in S} u^v(j) - g^v(N) = \sum_{j \in S} u^v(j) - g^v(S) = v(S) ,$$

so the game v_k is equal to the convex game v and hence, the convex game v itself is k-convex. In summary, a convex n-person game is k-convex if and only if all coalitions with at least k players are equally discontented in the game, i.e.,

$$g^v(S) = g^v(N) \quad \text{for all } S \subset N \text{ with } |S| \geq k \ . \tag{2.7}$$

From this we derive that the k-convexity of a convex n-person game generates the m-convexity for all $k \leq m \leq n - 2$. Without going into technical details, we remark that k-convexity and m-convexity where $k \neq m$ are in general contradictory notions for an arbitrary n-person game (cf. Driessen, 1988, page 181). Nevertheless, it turns out that the following three statements are equivalent.

1. The n-person game v is k-convex as well as $(k + 1)$-convex.
2. v is k-convex and $g^v(S) = g^v(N)$ for all $S \subset N$ with $|S| = k$.
3. v is $(k + 1)$-convex and $g^v(S) = g^v(N)$ for all $S \subset N$ with $|S| = k$.

3 The Bankruptcy Problem and k-Convexity of the Bankruptcy Game

We consider the bankruptcy problem of how to distribute the assets of the bankrupt entity among individuals according to their claims on it. As a case in point we mention the division problem of the fixed estate of a person who dies, leaving a number of fixed debts. The debts are supposed to be mutually inconsistent in that the estate is insufficient to meet all of the debts (otherwise the division problem is solved in such a way that all debts are completely met).

A **bankruptcy problem** is an ordered pair $(E; d)$, where $E \in \mathbb{R}$ and $d = (d_1, d_2, \ldots, d_n) \in \mathbb{R}^n$ such that $d_i > 0$ for all $1 \leq i \leq n$ and $0 < E < \sum_{j=1}^{n} d_j$. The positive real number E represents the estate and the positive real number d_i is the claim of creditor i on the estate. In order to formulate the bankruptcy problem as a game theoretic problem, O'Neill (1982) regarded the n creditors as players and described the worth of coalition S by taking into account the amount what is left of the estate E after each member i of the complementary coalition $N - S$ is paid the associated claim d_i. To be exact, the worth is equal to the remaining part of the estate or zero, whichever is more.

Definition 3.1: (cf. O'Neill, 1982). Let $(E; d)$ be a bankruptcy problem and let $N = \{1, 2, \ldots, n\}$ be the set of the n creditors. The corresponding **bankruptcy game** $v_{E;d}$ with player set N is given by

$$v_{E;d}(S) := \max\left[0, E - \sum_{j \in N-S} d_j\right] \quad \text{for all } S \subset N . \tag{3.1}$$

Due to $v_{E;d}(N) = E$, the game theoretic division problem of the total savings $v_{E;d}(N)$ of the grand coalition among the players agrees with the real division problem of the estate E among the creditors. It is known that bankruptcy games are convex games (cf. Curiel et al., 1987, Theorem 1). Our main goal is to present the conditions on the estate and the debts under which bankruptcy games are k-convex n-person games. For that purpose, we first treat a characterization of the condition (2.7) applied to bankruptcy games.

Lemma 3.2: Let $v_{E;d}$ be the bankruptcy game of (3.1) and let $S \subset N$ be such that $|S| \leq n - 2$. Then

$$g^{v_{E;d}}(S) = g^{v_{E;d}}(N) \quad \text{iff } E \geq \sum_{j \in N-S} d_j .$$

Proof: For the sake of notation, we write w instead of $v_{E;d}$. Due to (2.4) and (3.1), the utopia function $u^w \colon N \to \mathbb{R}$ is given by $u^w(i) = \min[E, d_i]$ for all $i \in N$. Let $S \subset N$ be such that $|S| \leq n - 2$. We mention the following straightforward equivalences: $g^w(S) = g^w(N)$ iff

$$\sum_{j \in N-S} u^w(j) = E - \max\left[0, E - \sum_{j \in N-S} d_j\right] \quad \text{iff}$$

$$\sum_{j \in N-S} \min[E, d_j] = \min\left[E, \sum_{j \in N-S} d_j\right] . \tag{3.2}$$

It remains to prove that (3.2) is equivalent to $E \geq \sum_{j \in N-S} d_j$.

(i) If $E \geq \sum_{j \in N-S} d_j$, then it follows immediately that $E \geq d_j$ for all $j \in N - S$ and thus, it is evident that (3.2) holds.

(ii) In order to prove the converse statement, suppose that (3.2) holds. We always have $|N - S| \geq 2$ and $0 < \min[E, d_j] \leq d_j$ for all $j \in N$. In case there would exist $i \in N - S$ with $E < d_i$, then we would obtain the strict inequalities $E < \sum_{j \in N-S} \min[E, d_j] < \sum_{j \in N-S} d_j$, but this result is in contradiction with (3.2). Now we conclude that $E \geq d_i$ for all $i \in N - S$. In view of this, (3.2) reduces to $\sum_{j \in N-S} d_j = \min\left[E, \sum_{j \in N-S} d_j\right]$ and consequently, $E \geq \sum_{j \in N-S} d_j$. This completes the proof of the lemma. □

Theorem 3.3: Let $k \in \mathbb{N}$, $1 \leq k \leq n - 2$. The bankruptcy game $v_{E;d}$ of (3.1) is a k-convex n-person game if and only if

$$E \geq \sum_{j \in S} d_j \quad \text{for all } S \subset N \text{ with } |S| = n - k . \tag{3.3}$$

Proof: We combine condition (2.7) and lemma 3.2 applied to the convex bankruptcy game $v_{E;d}$. As a result, the game $v_{E;d}$ is k-convex if and only if

$$E \geq \sum_{j \in N-S} d_j \quad \text{for all } S \subset N \text{ with } k \leq |S| \leq n - 2$$

or equivalently,

$$E \geq \sum_{j \in T} d_j \quad \text{for all } T \subset N \text{ with } 2 \leq |T| \leq n - k .$$

Since $d_i > 0$ for all $i \in N$, it suffices to satisfy the last constraints for the $(n - k)$-person coalitions. □

According to Theorem 3.3, the bankruptcy game is a k-convex n-person game where $1 \leq k \leq n - 2$ if and only if the estate is sufficient to meet the claims of creditors in any $(n - k)$-person coalition.

How to exploit the k-convexity property for the bankruptcy game? In general, player i's payoff by any core-element of an arbitrary game v is bounded above by his utopia payoff $u^v(i)$. Involving the bankruptcy game, player i's utopia payoff equals the minimum of his claim and the estate and in particular, his utopia payoff equals his claim whenever (3.3) holds. Without going into technical details, we remark that the k-convexity of the bankruptcy game, where $1 \leq k \leq n-2$, yields that the claim of all the creditors of any $(n-k)$-person coalition can be met by some core-element of the bankruptcy game, i.e., for all $S \subset N$ with $|S| = n - k$ there exists $x \in \mathrm{Core}(v_{E;d})$ such that $x_i = d_i$ for all $i \in S$. As an example, we treat the next problem which has already been studied in the Babylonian Talmud (cf. O'Neill, 1982).

Example 3.4: "Jacob died and each of his four sons Reuben, Simeon, Levi and Judah respectively produced a deed that Jacob willed to him his entire estate, half, one third, one quarter of his estate on his death. All deeds bear the same date and the total estate is 120 units".

The corresponding bankruptcy problem with the estate $E = 120$ and the four claims $d_1 = 30$, $d_2 = 40$, $d_3 = 60$, $d_4 = 120$, can be modelled as the 4-person bankruptcy game v given by

$$\begin{array}{lll} v(\{1,4\}) = 20, & v(\{1,2,4\}) = 60, & v(N) = 120\ , \\ v(\{2,4\}) = 30, & v(\{1,3,4\}) = 80, & v(S) = 0 \text{ otherwise}\ . \\ v(\{3,4\}) = 50, & v(\{2,3,4\}) = 90\ , & \end{array}$$

This 4-person bankruptcy game v is convex, but it is neither 1-convex nor 2-convex. Notice that the related bankruptcy problem with a variable estate E and the four fixed claims $d_1 = 30$, $d_2 = 40$, $d_3 = 60$, $d_4 = 120$, gives rise to a 4-person bankruptcy game which turns out to be 1-convex iff $220 \leq E < 250$ and further, it is 2-convex iff $180 \leq E < 250$. Finally, we mention that the idea of fixed claims and a variable estate plays a prominent part in Aumann and Maschler's approach to the bankruptcy problem (cf. Aumann and Maschler, 1985).

4 The Information Trading Problem and k-Convexity of the Information Market Game

We consider the information trading problem of how to sell and purchase the license rights of a new technology which is indispensable for the production of a new commodity. The information about the new technology is initially owned by a unique firm, the so-called patent holder. In the context of cooperative behaviour under perfect patent protection, the initially informed firm 1 seeks to sell the license rights of the new technology to all or some of the uninformed firms $2, 3, \ldots, n$. Let $N = \{1, 2, \ldots, n\}$ be the set of the n firms. The consumer market of the new commodity is supposed to be divided into submarkets on the basis of firms which have the right and possibility to enter such a submarket. For any coalition $T \subset N$, $T \neq \emptyset$, let $r_T \in \mathbb{R}$ represent the nonnegative maximal monetary profit obtainable from the production and sale of the new commodity in the submarket to which merely the firms in T may enter. For a detailed description of the information market situation in question, we refer to Muto, Potters and Tijs (1989).

The above model of an information market situation generates a cooperative n-person game with player set N and its characteristic function is interpreted as a profit function. There is no profit for coalitions consisting of uninformed firms only. Any coalition S containing the patent holder can attain the profit in each submarket to which at least one firm in S has access.

Definition 4.1: (cf. Muto, Potters and Tijs, 1989). Let $N = \{1, 2, \ldots, n\}$ be the set of the n firms of which the firm 1 is the patent holder and let $\{r_T \in \mathbb{R};\, T \subset N, T \neq \emptyset\}$ be the collection of nonnegative profits in submarkets. The corresponding **information market game** v with player set N is given by

$$
\begin{aligned}
v(S) &:= \sum_{T \subset N} [r_T;\, T \cap S \neq \emptyset] \quad && \text{for all } S \subset N \text{ with } 1 \in S \\
&= 0 && \text{for all } S \subset N - \{1\} \ . \qquad (4.1)
\end{aligned}
$$

Our aim is to present the conditions on the profits in the submarkets under which information market games are k-convex n-person games. The conditions involved are closely related to the nonexistence of profits with respect to a restricted class of submarkets to which the patent holder may not enter. To be exact, the information market game turns out to be 1-convex if and only if there is no profit obtainable from each submarket to which precisely one uninformed firm has access. Further, the convexity of the information market game is valid if and only if there is no profit obtainable from each submarket to which at least two uninformed firms may enter (excluding the patent holder). Finally, the information market game is a k-convex n-person game where $2 \leq k \leq n - 2$ if and only if there is no profit at all in each submarket to which the patent holder has no access.

Theorem 4.2: Let v be the information market game of (4.1).

(i) v is a convex n-person game iff $r_T = 0$ for all $T \subset N - \{1\}$ with $|T| \geq 2$.
(ii) v is a 1-convex n-person game iff $r_{\{i\}} = 0$ for all $i \in N - \{1\}$.
(iii) Let $k \in \mathbb{N}$, $2 \leq k \leq n - 2$. Then v is a k-convex n-person game iff $r_T = 0$ for all $T \subset N - \{1\}$.

Proof: For the sake of notation, we write r_i instead of $r_{\{i\}}$ for all $i \in N$ and moreover, put $r_\varnothing := 0$. Due to (2.4), (2.5) and (4.1), the utopia function $u^v \colon N \to \mathbb{R}$ and the gap function $g^v \colon \mathcal{N} \to \mathbb{R}$ of the information market game v are given by

$$u^v(1) = v(N) = \sum_{T \subset N} r_T, \qquad u^v(i) = r_i \qquad \text{for all } i \in N - \{1\} \ ,$$

$$g^v(N) = \sum_{j \in N - \{1\}} r_j, \qquad g^v(S) = \sum_{j \in S} r_j \quad \text{for all } S \subset N - \{1\} \ ,$$

$$g^v(S) = \sum_{j \in S - \{1\}} r_j + \sum_{T \subset N} [r_T; T \cap S = \varnothing] \quad \text{for all } S \subset N \text{ with } 1 \in S \ .$$

Especially, we observe that for all $S \subset N$ with $1 \in S$

$$g^v(S) = g^v(N) + \sum_{T \subset N} [r_T; T \cap S = \varnothing, |T| \geq 2] \ . \tag{4.2}$$

First of all, we establish the necessity of the conditions for the k-convexity of the information market game v.

(i) Suppose that the game v is convex. As noted in Section 2, the supermodularity of v yields the monotonicity of g^v and in particular, $g^v(N) \geq g^v(\{1\})$. In view of (4.2) applied to $S = \{1\}$, the inequality $g^v(N) \geq g^v(\{1\})$ reduces to $\sum_{T \subset N} [r_T; T \subset N - \{1\}, |T| \geq 2] \leq 0$ and consequently, $r_T = 0$ for all $T \subset N - \{1\}$ with $|T| \geq 2$.
(ii) Suppose that the game v is 1-convex. By (2.6), the 1-convexity of v implies that $g^v(\{i\}) \geq g^v(N)$ for all $i \in N - \{1\}$ or equivalently, $\sum_{j \in N - \{1, i\}} r_j \leq 0$ for all $i \in N - \{1\}$. From this we conclude that $r_i = 0$ for all $i \in N - \{1\}$.
(iii) Suppose that the game v is k-convex where $k \in \mathbb{N}$, $2 \leq k \leq n - 2$. By (2.6), the k-convexity of v implies that $g^v(N - \{1, i\}) \geq g^v(N)$ for all $i \in N - \{1\}$ or equivalently, $r_i = 0$ for all $i \in N - \{1\}$. Now it remains to show that $r_T = 0$ for all $T \subset N - \{1\}$ with $|T| \geq 2$ or equivalently, $g^v(N) \geq g^v(\{1\})$ (the equivalence holds as seen at the end of part (i)). As noted beneath Definition 2.2, we have the equalities $g^v(N) = g^{v_k}(N)$, $g^v(\{1\}) = g^{v_k}(\{1\})$, and in addition, the inequality $g^{v_k}(N) \geq g^{v_k}(\{1\})$ holds because of the convexity of the game v_k. It follows that the inequality $g^v(N) \geq g^v(\{1\})$ holds as was to be shown. This completes the proof concerning the necessity of the

conditions for the k-convexity of the game v. Next we prove that the conditions involved are sufficient for the k-convexity of v.

(iv) Suppose that either $k = 1, r_i = 0$ for all $i \in N - \{1\}$ or $2 \leq k \leq n-2, r_T = 0$ for all $T \subset N - \{1\}$. We obtain $u^v(1) = v(N)$, $u^v(i) = r_i = 0$ for all $i \in N - \{1\}$ and $g^v(N) = 0$. Evidently, the condition (2.6) holds since $g^v(S) \geq 0$ for all $S \subset N$. In view of Definition 2.2, it is straightforward to verify that the n-person game v_k is given by

$$\begin{aligned} v_k(S) &= v(N) \quad \text{for all } S \subset N \text{ with } 1 \in S \\ &= 0 \qquad\quad \text{for all } S \subset N - \{1\} \ . \end{aligned}$$

From this we directly conclude that the game v_k satisfies the convexity condition (2.1) applied to v_k and therefore, the game v is k-convex.

(v) Suppose that $r_T = 0$ for all $T \subset N - \{1\}$ with $|T| \geq 2$. Put $\beta := \sum_{T \subset N} [r_T; |T| \geq 2, 1 \in T]$. Now we obtain that

$$\begin{aligned} v(S) &= \sum_{j \in S} r_j + \beta \quad \text{for all } S \subset N \text{ with } 1 \in S \\ &= 0 \qquad\qquad\quad \text{for all } S \subset N - \{1\} \ . \end{aligned}$$

Now it follows immediately that the game v satisfies the convexity condition (2.1).

In summary, we have established that the conditions involved are necessary and sufficient for the k-convexity of the information market game v. This completes the proof of the theorem. □

Notice that for information market games, the k-convexity property where $2 \leq k \leq n-2$ agrees with the combination of the convexity and 1-convexity properties. The general situation in which k-convexity implies 1-convexity and/or convexity is rather extraordinary. Moreover, the combination of the convexity and 1-convexity properties can be characterized by a constant gap function (i.e., $g^v(S) = g^v(N) \geq 0$ for all $S \subset N, S \neq \emptyset$).

How to exploit the k-convexity properties for the information market game? It turns out that the core of a k-convex information market game v where $1 \leq k \leq n-2$ degenerates into a singleton whose unique point $(v(N), 0, 0, \ldots, 0)$ equals the τ-value and the nucleolus, that is no payoff to uninformed firms. If the information market game v happens to be convex, then it can be shown that both the τ-value and the nucleolus coincide with the average of the individual worth vector $(v(\{1\}), 0, 0, \ldots, 0)$ and the utopia payoff vector $(v(N), r_{\{1\}}, r_{\{2\}}, \ldots, r_{\{n\}})$, that is each uninformed firm receives half of its own individual profit.

In this paper we have studied the k-convexity properties for two classes of games, namely bankruptcy games and information market games. Concerning studies of k-convexity properties for other classes of games, we refer to Driessen (1991) which treats the class of clan games, and Driessen (1994).

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